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EXTENSIONS OF SHEAVES OF COMMUTATIVE ALGEBRAS BY NONTRIVIAL KERNELS

by

D. H. Van Osdol*

Introduction. Let X be a topological space, R a sheaf of commutative algebras on X, and A a sheaf of R-modules considered as an algebra with trivial multiplication. It was shown in [5] that the group of equivalence classes of commutative algebra extensions of R with A as kernel is isomorphic to $H^1(R,A)$, the first bicohomology group of R with coefficients in A. In this paper we will not assume that A has trivial multiplication; we will find that, if ZA is the center of A, then $H^2(R,ZA)$ contains all of the obstructions to the existence of extensions of R by A which "realize" a given morphism. This will generalize the results of [1] to the category of sheaves, and of [4] in that no assumptions need be made on X or R.

In order to keep this paper as short as possible, we shall follow the format of [1]. We shall not, however, generalize section 4 of [1]. There are two reasons for this: first, we do not know how to globalize Barr's theory, although we can do his section 4 locally using only triple-theoretic techniques (and then the underlying set of A is $Z \times K$ where K is the kernel of R's structure morphism); secondly, the correct setting for completely characterizing the bicohomology H^n, $n>1$, will not be known until Duskin writes up his results [3].

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Let Sets be the category of pointed sets. The distinguished point of a set will be the zero of any corresponding algebra. Let Λ be a sheaf of commutative rings on X, $F(X, \mathrm{Alg})$ the category of sheaves of commutative Λ-algebras on X, $\Pi\Lambda_x$-alg the product over $x \in X$ of the categories of Λ_x-algebras (Λ_x = stalk of Λ at $x \in X$), and $F(X, \mathrm{Sets})$ the category of sheaves of pointed sets. We should stress that our algebras need not have unit elements. It is easy to verify that we have a bicohomology situation [5] :

$$\begin{tikzcd}
F(X, \mathrm{Alg}) \arrow[r, shift left, "S"] \arrow[d, shift left, "U"] & \Pi\Lambda_x\text{-alg} \arrow[l, shift left, "Q"] \arrow[d, shift left, "U"] \\
F(X, \mathrm{Sets}) \arrow[r, shift left, "S"] \arrow[u, shift left, "F"] & \mathrm{Sets}\,|x| \arrow[l, shift left, "Q"] \arrow[u, shift left, "F"]
\end{tikzcd}$$

where the horizontal arrows are adjoint resolutions of the Godement standard construction, and the vertical ones are the obvious free and forgetful functors. Given a sheaf R of Λ-algebras and a sheaf Z of R-modules, the bicohomology theory we use is that arising from the above picture and the functor Der_Λ. Hence we take a "free" simplicial resolution of R, a Godement cosimplicial resolution of Z, and examine the cohomology groups of the double complex gotten by looking at Λ-derivations of the resolution over R into the resolution under Z.

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I. The Class E

There is no problem in globalizing section 1 of [1] , but we will give a brief outline in order to fix notation. Let A be a sheaf of ideals in C and for each $x \in X$ let $Z(A_x, C_x) = \{c \in C_x | cA_x = 0\}$. Define the centralizer of A in C to be the pullback

$$\begin{array}{ccc} Z(A,C) & \longrightarrow & Q\{Z(A_x,C_x) | x \in X\} \\ \downarrow & & \downarrow \\ C & \xrightarrow{\eta C} & QSC \end{array} ,$$

and the center of A to be $ZA = Z(A,A)$. Then $Z(A,C)$ is a sheaf of ideals in C and we let $E(A)$ denote the set of equivalence classes of exact sequences of sheaves of commutative algebras

$$0 \to ZA \to A \to C/Z(A, C) \to C/A + Z(A, C) \to 0 .$$

Here equivalence is by isomorphisms which fix ZA and A .

On the other hand, let E be any sheaf of subalgebras of the sheaf of germs of endomorphisms of A such that E contains the image of $\lambda: A \to \mathrm{Hom}_A(A,A)$. For each $a \in A$ and open U in X , $\lambda U(a): A|_U \to A|_U$ is defined by $[\lambda U(a)]V(a') = [A(i)a] \cdot a'$ where i is the inclusion of V in U , $a' \in A(V)$, and "$\cdot$" represents multiplication. Let E' be the set of all such E .

Proposition 1.1. There is a natural one-one correspondence $E \cong E'$.

Proof. As in [1] . Here we also construct the truncated simplicial algebra

$$B \underset{d^2}{\overset{d^0}{\Rrightarrow}} P \underset{d^1}{\overset{d^0}{\rightrightarrows}} E \xrightarrow{\pi} M , \qquad s^0 : E \to P .$$

Proposition 1.2. The above simplicial algebra is exact.

Proposition 1.3. There is a derivation $\delta : B \to ZA$ given by $\delta = (P - s^o . d^o).(d^o - d^1 + d^2)$.

II. The Obstruction to a Morphism.

Let R be a sheaf of commutative algebras, $p : R \to M$ a surjection, and $0 \to A \to C \to R \to 0$ an exact sequence (extension) of commutative algebras. We say that p arises from this extension if there is a commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[r] & A \arrow[r] \arrow[d, "A"] & C \arrow[r] \arrow[d, "\nu_o"] & R \arrow[r] \arrow[d, "p"] & 0 \\
0 \arrow[r] & ZA \arrow[r] & A \arrow[r] & E \arrow[r, "\pi"] & M \arrow[r] & 0
\end{tikzcd}$$.

Given a surjection p , we wish to determine if there are any extensions from which it arises.

Since $\pi : E \to M$ is surjective, there is a map $s : SUM \to SUE$ such that $SU\pi . s = SUM$. By adjointness we get $s' : FUM \to QSE$ such that the diagram

$$\begin{tikzcd}
FUM \arrow[r, "s'"] \arrow[d, "\epsilon M"'] & QSE \arrow[d, "QS\Pi"] \\
M \arrow[r, "\eta M"] & QSM
\end{tikzcd}$$

commutes. Let $p_o = s'.FUp$. Then

$$QS\pi . p_o . \epsilon FUR = \eta M . p . \epsilon R . \epsilon FUR$$
$$= \eta M . p . \epsilon R . FU\epsilon R$$
$$= QS\pi . p_o . FU\epsilon R$$

so there exists a unique $\tilde{p}_1 : (FU)^2 R \to QS\tilde{P}$ such that $QS\tilde{d}^o . \tilde{p}_1 = p_o . \epsilon FUR$, $QS\tilde{d}^1 . \tilde{p}_1 = p_o . FU\epsilon R$. Here $(\tilde{P}, \tilde{d}^i)$ is the kernel pair of π , and QS preserves finite limits. Now the unique map $u : P \to \tilde{P}$ such that $\tilde{d}^i . u = d^i$ is surjective, so there is $t : SU\tilde{P} \to SUP$ splitting it. Using this map and adjointness we produce $t' : FUQS\tilde{P} \to (QS)^2 P$ such that $(QS)^2 u . t' = \eta QS\tilde{P} . \epsilon QS\tilde{P}$.

Define $\bar{p}_1:(FU)^3R \to (QS)^2P$ by $\bar{p}_1 = t'.FU\tilde{p}_1$ and then $p_1 = \mu P.\bar{p}_1.\delta' GR$ where μ = multiplication for QS, δ'= comultiplication for FU. One computes that $QSu.p_1 = \tilde{p}_1$, from which it follows that there is a unique $p_2:(FU)^3R \to QSB$ such that $d^i.p_2 = p_1.\epsilon^i$, $0 \leq i \leq 2$ where $\epsilon^i = FU^i\,\epsilon FU^{2-i}R$. By the naturality of ϵ, $T\delta.p_2.\Sigma_{i=o}^3(-1)^i\,\epsilon^i = 0$.

On the other hand,

$$
\begin{aligned}
(QS)^2\,\pi.\eta QSE.p_o &= \eta QSM.QS\pi.p_o \\
&= \eta QSM.\eta M.p.\epsilon R \\
&= QS\eta M.\eta M.p.\epsilon R \\
&= QS\eta M.QS\pi.p_o \\
&= (QS)^2\,\pi.QS\eta E.p_o
\end{aligned}
$$

so there is a unique $\tilde{q}_1:FUR \to (QS)^2\tilde{P}$ such that $(QS)^2\tilde{d}^i.\tilde{q}_1 = \eta^iE.p_o$, $i = 0, 1$, where η^iE is defined as was ϵ^i above. Let as before $t'':FU(QS)^2\tilde{P} \to (QS)^3P$ be such that $(QS)^3u.t'' = \eta(QS)^2\tilde{P}.\epsilon(QS)^2\tilde{P}$. Define $\bar{q}_1 = t''.FU\tilde{q}_1$ and $q_1 = \mu QSP.\bar{q}_1.\delta' R$. Then $(QS)^2u.q_1 = \tilde{q}_1$ and q_1 induces $q_2:FUR \to (QS)^3B$ such that $(QS)^3d^i = \eta^iP.q_1$, $0 \leq i \leq 2$. The induced derivation $(QS)^3\delta.q_2$ has the property that $\Sigma_{i=o}^3(-1)^i\eta^i.T^3\delta.q_2 = 0$.

Finally, for $i = 0, 1$ consider $(QS)^2\tilde{d}^i.\eta^i.\tilde{p}_1:(Fu)^2R \to (QS)^2E$. One computes that $(QS)^2\pi.(QS)^2\tilde{d}^o.\eta QS\tilde{P}.\tilde{p}_1 = (QS)^2\pi.(QS)^2\tilde{d}^1.QS\eta\tilde{P}.\tilde{p}_1$ and concludes that there exists $\tilde{v}:(FU)^2R \to (QS)^2\tilde{P}$ such that $(QS)^2\tilde{d}^i.\tilde{v} = (QS)^2\tilde{d}^i.\eta^i.\tilde{p}_1$ for $i=0, 1$. As before, the fact that $u:P \to \tilde{P}$ is surjective allows us to define $v:(FU)^2R \to (QS)^2P$ such that $(QS)^2u.v = \tilde{v}$. Let $r_1:(FU)^2R \to (QS)^2B$ be the unique map such that $(QS)^2d^o.r_1 = \eta QSP.p_1$, $(QS)^2d^1.r_1 = v$, $(QS)^2d^2.r_1 = q_1.FU\epsilon R$ (it is easy to see that such r_1 exists, because $(QS)^2B$ is the kernel triple of $(QS)^2d^o$ and $(QS)^2d^1$). Similarly let $(QS)^2d^o.r_2 = q_1.\epsilon FUR$, $(QS)^2d^1.r_2 = v$, and $(QS)^2d^2.r_2 =$

$QS\eta P.p_1$. Now we have:

$$((QS)^2\delta.r_2-(QS)^2\delta.r_1).\Sigma_{i=o}^{2}(-1)^i\epsilon^i$$

$$= (QS)^2(P-s^o.d^o).(q_1.\epsilon^o-v+QS\eta P.p_1).\Sigma_{i=o}^{2}(-1)^i\epsilon^i$$
$$-(QS)^2(P-s^o.d^o).(\eta QSP.p_1-v+q_1.\epsilon^1).\Sigma_{i=o}^{2}(-1)^i\epsilon^i$$

$$= -(QS)^2(P-s^o.d^o).(\Sigma_{j=o}^{1}(-1)^j\eta^j).p_1.(\Sigma_{i=o}^{2}(-1)^i\epsilon^i)$$

$$= (\Sigma_{j=o}^{1}(-1)^j\eta^j).QS\delta.p_2 ,$$

and similarly $(\Sigma_{j=o}^{2}(-1)^j\eta^j).((QS)^2\delta.r_2-(QS)^2\delta.r_1)$

$$= (QS)^3\delta.q_2.\Sigma_{i=o}^{1}(-1)^i\epsilon^i .$$

Hence $(QS\delta.p_2 , (QS)^2\delta.r_2-(QS)^2\delta.r_1 , (QS)^3\delta.q_2)$ is a cocycle in the bicohomology double complex; we will denote its cohomology class by $[p]$ and call $[p]$ the obstruction of $[p]$. We say $[p]$ is unobstructed if $[p] = 0$. This teminology is justified by the next two results.

Proposition 2.1. The cohomology class of $(QS\delta.p_2, (QS)^2\delta.r_2-(QS)^2\delta.r_1 , (QS)^3\delta.q_2)$ is independent of the choices of $s:SUM \to SUE$ and $t:SU\tilde{P} \to SUP$.

Proof. Once we have p_1 , q_1 , and v the maps p_2 , q_2 , r_1 , and r_2 are uniquely determined. So suppose σ_o , σ_1 , τ_1 , ρ_1 , ρ_2 ,are different choices of p_o , p_1 , q_1 , r_1 , r_2 and construct simplicial homotopies as in [1] . Specifically let $QS\tilde{d}^o.\tilde{h}^o = p_o$, $QS\tilde{d}^1.\tilde{h}^o = \sigma_o$, $Tu.h^o = \tilde{h}^o$, and $QSd^o.v' = QSd^o.p_1$, $QSd^1.v' = QSd^1.\sigma_1$. Considering the maps p_1 , v' , and $h^o.\epsilon^1$ from $(FU)^2R$ to QSP we see that there exists $h^o:(FU)^2R \to QSB$ such that $QSd^o.h^o = p_1$, $QSd^1.h^o = v'$, and $QSd^2.h^o = h^o.\epsilon^1$. Similary there exists $h^1:(FU)^2R \to QSB$ such that $QSd^O.h^1 = h^o.\epsilon^O$, $QSd^1.h^1 = v'$, and $QSd^2.h^1 = \sigma_1$. From these relations it is easy to compute that $(QS\delta.h_O-QS\delta.h_1).\Sigma_{i=o}^{2}(-1)^i\epsilon^i = QS\delta.p_2 -QS\delta.\sigma_2$.

Now let $w: FUR \to (QS)^2P$ be such that $(QS)^2d^o.w = (QS)^2d^o.q_1$ and $(QS)^2d^1.w = (QS)^2d^1.\tau_1$ where τ_1 "lifts" σ_o. As above let $k^o, k^1: FUR \to (QS)^2B$ be determined by the conditions $(QS)^2d^o.k^o = q_1$, $(QS)^2d^1.k^o = w$, $(QS)^2d^2.k^o = QS\eta P.h^o$, $(QS)^2d^o.k^1 = \eta QSP.h^o$, $(QS)^2d^1.k^1 = w$, and $(QS)^2d^2.k^1 = \tau_1$. Again one finds that $(\Sigma_{j=o}^2(-1)^j.\eta^j).((QS)^2\delta.k^o - (QS)^2\delta.k^1)$ $= (QS)^3\delta.q_2 - (QS)^3\delta.\tau_2$. Finally,
$((QS)^2\delta.k^o - (QS)^2\delta.k_1).\Sigma_{i=o}^1(-1)^i\varepsilon^i - (\Sigma_{j=o}^1(-1)^j\eta^j).(QS\delta.h^o - QS\delta.h^1)$
$= (QS)^2\delta.\rho_1 - (QS)^2\delta.\rho_2 - (QS)^2\delta.r_1 + (QS)^2\delta.r_2$. Hence the chomology class of $(QS\delta.p_2, (QS)^2\delta.r_2 - (QS)^2\delta.r_1, (QS)^3\delta.q_2)$ agrees with that of $(QS\delta.\sigma_2, (QS)^2\delta.\rho_2 - (QS)^2\delta.\rho_1, (QS)^3\delta.\tau_2)$, as was to be shown.

Theorem 2.2. A surjection $p: R \to M$ arises from an extension if and only if p is unobstructed.

Proof. Suppose p arises from an extension $0 \to A \to C \xrightarrow{\theta} R \to 0$ and let K be the kernel pair of θ.
Then we have a commutative diagram:

$$\begin{tikzcd}
K \arrow[r, shift left, "e^o"] \arrow[r, shift right] \arrow[d, "\nu_1"] & C \arrow[l, bend left=60, "e^1"', "t^o"] \arrow[r, "\theta"] \arrow[d, "\nu_o"] & R \arrow[r] \arrow[d, "p"] & 0 \\
P \arrow[r, shift left] \arrow[r, shift right] & E \arrow[l, shift left=2] \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$.

Moreover we can find $\sigma_o: FUR \to QSC$ such that $QS\theta.\sigma_o = \eta R.\varepsilon R$. If we let $\sigma_1: (FU)^2R \to QSK$ be such that $QSe^i.\sigma_1 = \sigma_o.\varepsilon^i$ and $\tau_1: FUR \to (QS)^2K$ such that $(QS)^2e^i.\tau_1 = \eta^i.\sigma_o$ for $i=0, 1$ then $QS\nu_o.\sigma_o$ serves as p_o, $QS\nu_1.\sigma_1$ as p_1, and $(QS)^2\nu_1.\tau_1$ as q_1. By 2.1 we can assume that things have been so arranged. But then using the fact that $(QS)^je^o$, $(QS)^je^1$ is a kernel pair for each $j \geq 0$, one can show that $QS(K-t^o.e^o).\sigma_1.\Sigma_{i=o}^2(-1)^i\varepsilon^i = 0$,

$(QS)^2(K-t^o.e^o).\left[(\Sigma_{j=o}^1(-1)^j\eta^j).\sigma_1 - \tau_1.(\Sigma_{j=o}^1(-1)^i\epsilon^i)\right] = 0$, and $(QS)^3(K-t^o.e^o).(\Sigma_{j=o}^2(-1)^j\eta^j).\tau_1 = 0$. From this it follows that $QS\delta.p_2 = 0$, $(QS)^2\delta.r_2 - (QS)^2\delta.r_1 = 0$, and $(QS)^3\delta.q_2 = 0$. Thus $[p] = 0$.

Conversely, suppose $[p] = 0$. Then there exist $\tau:(FU)^2R \to QS(ZA)$, $\rho:FUR \to (QS)^2ZA$ with $\tau.\epsilon = QS\delta.p_2$, $\eta.\rho = (QS)^3\delta.q_2$, and $\rho.\epsilon - \eta.\tau = (QS)^2\delta.r_2 - (QS)^2\delta.r_1$. Here we abbreviate $\Sigma_{i=o}^n(-1)^i\epsilon^i = \epsilon$ and similarly for η. Now $\bar{p}_1 = p_1 - \tau$, $\bar{q}_1 = q_1 - \rho$ serve as new p_1, q_1 and also give $\bar{p}_2$, $\bar{q}_2$, $\bar{r}_1$, $\bar{r}_2$. We have

$$\begin{aligned} QS(P - s^o.d^o).QSd.\bar{p}_2 &= QS(P - s^o.d^o).\bar{p}_1.\epsilon \\ &= QS(P - s^o.d^o).p_1.\epsilon - QS(P-s^o.d^o).\tau.\epsilon \\ &= QS(P - s^o.d^o).p_1.\epsilon \\ &\quad - \tau.\epsilon + QSs^o.QSd^o.\tau.\epsilon \\ &= QS\delta.p_2 - \tau.\epsilon \\ &= 0 \end{aligned}$$

because the kernel of QSd^o is $QS(Z(A,P))$ which contains $QS(ZA)$. Similar computations yield $(QS)^3(P-s^o.d^o).(QS)^3d.\bar{q}_2 = 0$ and $(QS)^2(P-s^o.d^o).(QS)^2d.\bar{r}_2 - (QS)^2(P-s^o.d^o).(QS)^2d.\bar{r}_1 = 0$. Hence we can assume that $(QS)^3\delta.q_2$, $QS\delta.p_2$, $(QS)^2\delta.r_2 - (QS)^2\delta.r_1$ are all zero (by Prorosition 2.1). We now go over to the equivalent category $(\text{Sets}^{|X|})_G^T$ where $T = UF$, $G = SQ$. The reader is referred to [6] for a clarification of what this means, and to [5] for an introduction to the techniques to be used below. Let R, M, E, $\tilde{P}$, P, B, A, ZA be translated respectively into $\{R_x, \xi_1, \xi_2\}$, $\{M_x, \beta_1, \beta_2\}$, $\{E_x, \gamma_1, \gamma_2\}$, $\{\tilde{P}_x, \bar{\nu}_1, \bar{\nu}_2\}$, $\{A_x x E_x, \nu_1, \nu_2\}$, $\{B_x, -, -\}$ $\{A_x, \alpha_1, \alpha_2\}$, $\{ZA_x, -, -\}$. Since we want to use the symbols p_i for projections from a product, we let our old p_i be u_i, $0 \leq i \leq 2$.

For notational convenience, we drop all subscripts x and say once and for all that an equation will stand for the same equation with subscripts adjoined. For example, $\theta.s = M$ means $\theta_x.s_x = M_x$ for each x in X. Our assumption that $(QS)^3\delta.q_2$ e.t.c. are all zero translates into the following three equations in $(\mathrm{Sets}^{|X|})^T_G$:

i) $p_1.u_1.T\xi_1 - p_1.u_1.\mu R + p_1.\nu_1.Tu_1. = 0$

ii) $G^2p_1.G\nu_2.q_1 - G^2p_1.\delta'(AxE).q_1 + G^2p_1.Gq_1.\xi_2 = 0$

iii) $Gp_1.q_1.\xi_1 - Gp_1.G\nu_1.\lambda P.Tq_1 = Gp_1.Gu_1.\lambda R.T\xi_2 - Gp_1.\nu_2.u_1$

Here $\lambda:TG \to GT$ is the distributive law (see [5]) , and p_1 (or p_2) is the first (or second) projection from the appropriate product. Since our presentation has now begun to differ significantly from that of Barr [1] , we will provide more detail than earlier in the paper. Let $C = AxR$, and define $\zeta_1:TC \to C$, $\zeta_2:C \to GC$ by the conditions $p_1.\zeta_1 = p_1.\nu_1.T(p_1xs.p.p_2) + p_1.u_1.Tp_2$, $p_2.\zeta_1 = \xi_1.Tp_2$, $Gp_1.\zeta_2 = \alpha_2.p_1 + Gp_1.q_1.p_2$, $Gp_2.\zeta_2 = \xi_2.p_2$. We claim that (C , ζ_1 , ζ_2) is in $(\mathrm{Sets}^{|X|})^T_G$. Besides the "cocycle identities" listed above, the only fact we need is that $\nu_1:T(AxE) \to AxE$ has the following property: For each $g:X \to A$ and $f:X \to AxE$ we have

iv) $p_1.\nu_1.T([g+p_1.f]xd^1.f) = p_1.\nu_1.T(gxd^0.f) + p_1.\nu_1.Tf$.

Since this amounts to a combinatorial identity, we relegate its proof to the Appendix. Using i) and iv) we can prove that ζ_1 is associative:

$$p_1.\zeta_1.T\zeta_1 = [p_1.\nu_1.T(P_1xs.p.p_2) + p_1.u_1.Tp_2].T\zeta_2$$
$$= p_1.\nu_1.T([p_1.\nu_1.T(p_1xs.p.p_2) + p_1.u_1.Tp_2]xs.p.\xi_1.Tp_2)$$
$$+ p_1.u_1.T\xi_1.T^2p_2$$

$$= p_1.\nu_1.T([p_1.\nu_1.T(p_1xs.p.p_2)x\gamma_1.Ts.Tp.Tp_2) + p_1.\nu_1.Tu_1.T^2p_2 + p_1.u_1.T\xi_1.T^2p_2$$

$$= p_1.\nu_1.T(\nu_1.T(p_1xs.p.p_2)) + p_1.u_1.\mu R.T^2p_2$$

$$= p_1.\nu_1.\mu(AxE).T^2(p_1xs.p.p_2) + p_1.u_1.\mu R.T^2p_2$$

$$= p_1.\zeta_1.\mu(AxR) ;$$

the fact that $p_2.\zeta_1.T\zeta_1.T\zeta_1 = p_2.\zeta_1.\mu(AxR)$ is an easy computation. Notice that in the above computation we have taken $g = p_1.\nu_1.T(p_1xs.p.p_2)$ and $f = u_1.Tp_2$ in iv). Before proving that ζ_1 is unitary, we show that u_1 is "normalized":

$$0 = (p_1.u_1.T\xi_1 - p_1.u_1.\mu R + p_1.\nu_1.Tu_1).\eta TR$$

$$= p_1.u_1.\eta R.\xi_1 - p_1.u_1 + p_1.\nu_1.\eta R.u_1$$

$$= p_1.u_1.\eta R.\xi_1 \quad .$$

But composing this equation with ηR gives $p_1.u_1.\eta R = 0$, and from this it follows that ζ_1 is unitary:

$$\zeta_1.\eta(AxR) = [P_1.\nu_1. \quad T(p_1xs.p.p_2).\eta(AxR) + p_1.u_1.Tp_2.\eta(AxR)] x\xi_1.Tp_2.\eta(AxR)$$

$$= [p_1.([p_1xs.p.p_2] + p_1.u_1.\eta R.T^2p_2]x\xi_1.\eta R.T^2p_2$$

$$= p_1xp_2 \quad .$$

The computations which show that ζ_2 is counitary and coassociative use only ii) above, and will be omitted. The "compatibility" of ζ_1 and ζ_2 uses iii) and iv) above, and proceeds as follows:

$$Gp_1.G\zeta_1.\lambda(AxR).T\zeta_2 = G(p_1.\nu_1.T(p_1xs.p.p_2) + p_1.u_1.Tp_2).\lambda(AxR).T\zeta_2$$

$$= Gp_1.G\nu_1.GT(p_1xs.p.p_2).\lambda(AxR).T\zeta_2 + Gp_1.Gu_1.GTp_2.\lambda(AxR).T\zeta_2$$

$$= Gp_1.G\nu_1.\lambda(AxE).TG(p_1xs.p.p_2).T\zeta_2 + Gp_1.Gu_1.\lambda R.TGp_2.T\zeta_2$$

$$= Gp_1.G\nu_1.\lambda(AxE).T([\alpha_2.p_1 + Gp_1.q_1.p_2]xGs.Gp.\xi_2.p_2) + Gp_1.Gu_1.\lambda R.T(\xi_2.p_2)$$

$$= Gp_1.G\nu_1.\lambda(AxE).T(\alpha_2.p_1x\gamma_2.s.p.p_2) + Gp_1.G\nu_1.\lambda(AxE).Tq_1.Tp_2 + Gp_1.Gu_1.\lambda R.T\xi_2.Tp_2$$

$$= Gp_1.G\nu_1.\lambda(AxE).T\nu_2.T(p_1xs.p.p_2) + Gp_1.q_1.\xi_1.Tp_2 + Gp_1.\nu_2.u_1.Tp_2$$

$$= Gp_1 \cdot \nu_2 \cdot \nu_1 \cdot T(p_1 x s.p.p_2) + \alpha_2 \cdot p_1 \cdot u_1 \cdot Tp_2 + Gp_1 \cdot q_1 \cdot \xi_1 \cdot Tp_2$$
$$= \alpha_2 \cdot p_1 \cdot \zeta_1 + Gp_1 \cdot q_1 \cdot p_2 \cdot \zeta_1$$
$$= Gp_1 \cdot \zeta_2 \cdot \zeta_1 \; ;$$

here, again, that $Gp_2 \cdot G\zeta_1 \cdot \lambda(AxR) \cdot T\zeta_2 = Gp_2 \cdot \zeta_2 \cdot \zeta_1$ is obvious. Notice that we have not used iv) as it stands, but rather the analog of iv) for $GP = G(AxE)$. We have taken $g = \alpha_2 \cdot p_1$ and $f = q_1 \cdot p_2$. At any rate, (C, ζ_1, ζ_2) is in $(\mathrm{Sets}^{|X|})^T_G$ and the first injection, second projection give us an exact sequence $0 \to A \xrightarrow{i} AxR = C \xrightarrow{p_2} R \to 0$ in $(\mathrm{Sets}^{|X|})^T_G$.

Define $h: C \to E$ by $h = \lambda \cdot p_1 + s.p.p_2$. Clearly $\pi.h = p.p_2$ and $h.i = \lambda$, so that if h is a morphism in $(\mathrm{Sets}^{|X|})^T_G$ then we will have produced an extension from which p arises, and the proof will be complete. But we have:

$$
\begin{aligned}
h.\zeta_1 &= \lambda.(p_1 \nu_1 . T(p_1 x s.p.p_2) + p_1 . u_1 . Tp_2) + s.p.\xi_1 . Tp_2 \\
&= \lambda.p_1.\nu_1.T(p_1 x s.p.p_2) + \gamma_1.Ts.Tp.Tp_2 - s.p.\xi_1.Tp_2 + s.p.\xi_1.Tp_2 \\
&= \lambda.p_1.\nu_1.T(p_1 x s.p.p_2) + p_2.\nu_1.T(p_1 x s.p.p_2) \\
&= d^o.\nu_1.T(p_1 x s.p.p_2) \\
&= \gamma_1.Td^o.T(p_1 x s.p.p_2) \\
&= \gamma_1.T(\lambda.p_1 + s.p.p_2) \\
&= \gamma_1.Th \quad , \text{ and}
\end{aligned}
$$

$$
\begin{aligned}
Gh.\zeta_2 &= G\lambda.(\alpha_2.p_1 + Gp_1.q_1.p_2) + Gs.Gp.\xi_2.p_2 \\
&= \gamma_2.\lambda.p_1 + \gamma_2.s.p.p_2 - Gs.Gp.\xi_2.p_2 + Gs.Gp.\xi_2.p_2 \\
&= \gamma_2.h \quad .
\end{aligned}
$$

III. The Action of H^1

Theorem 3.1. Let $p: R \to M$ be unobstructed, and let Σ denote the equivalence classes of extensions of R by A which induce p. Then the group $H^1(R, ZA)$ acts on Σ as a principal homogeneous representation.

Proof. It is shown in [5] that $H^1(R, ZA)$ is in one-one correspondence with the set of equalence classes of singular extensions of R by ZA . Once this is known, Barr's proof of this proposition [1] translates almost verbatum into a proof for sheaves.

Appendix.

In this appendix we give a proof of equation iv) above (section III), and compare Barr's constructions [1] to our own. To dispose of equation iv), recall that given a commutative algebra A, its structure map $\alpha : TA \to A$ takes a polynomial in elements of A to the "value" of the polynomial. That is, α remembers that A is an algebra and uses the algebra operations in A to compute the polynomial. Now multiplication in $P = A \times E$ is defined by $(a_1, x_1)(a_2, x_2) = (a_1a_2 + x_1a_2 + a_1x_2, x_1x_2)$ where x_1a_2 and a_1x_2 denote the value of x on a.

Prosition A.1. Given $a_i \in A$, $x_i \in E$ for $1 \leq i \leq n$ we have $\prod_{i=1}^{n}(a_i, x_i) = (\Sigma f(1)_1 --- f(n)_n , x_1 --- x_n)$ where the sum is taken over all functions $f : \underline{n} = \{1, 2, ---, n\} \to \{a, x\}$ such that f is not identically equal to x.

Proof. By induction on n. We have

$$\prod_{i=1}^{n}(a_i , x_i) = (\Sigma f(1)_1 --- f(n-1)_{n-1} , x_1 --- x_{n-1})(a_n , x_n)$$
$$= (\Sigma f(1)_1 --- f(n-1)_{n-1}a_n + \Sigma f(1)_1 --- f(n-1)_{n-1}x_n + x_1 --- x_{n-1}a_n , x_1 --- x_n)$$
$$= (\Sigma f(1)_1 --- f(n)_n , x_1 --- x_n)$$

where the indexing sets for the sums are clear.

Proposition A.2 Given a_i, $b_i \in A$, $x_i \in E$ for $1 \leq i \leq n$ we have that $\prod_{i=1}^{n}(a_i + b_i , x_i)$ and $\prod_{i=1}^{n}(b_i , \lambda a_i + x_i) + \prod_{i=1}^{n}(a_i , x_i)$ have the same first coordinates.

Proof. Induction on n and Proposition A.1.

$$\prod_{i=1}^{n}(a_i+b_i\ ,\ x_i) = (\Sigma g(1)_1{-}{-}{-}g(n-1)_{n-1}+\Sigma h(1)_1{-}{-}{-}h(n-1)_{n-1}\ ,\ x_1{-}{-}{-}x_{n-1})\,(a_n+b_n\ ,\ x_n)$$

where the g's run through the set of functions from $\underline{n-1} \to \{b\ ,\ \lambda a+x\}$ which are not identically $\lambda a + x$ and the h's through all $\underline{n-1} \to \{a\ ,\ x\}$ which are not identically x.

Hence we get as first coordinate

$$\Sigma g(1)_1{-}{-}{-}g(n-1)_{n-1}a_n+\Sigma g(1)_1{-}{-}{-}g(n-1)_{n-1}b_n+\Sigma h(1)_1{-}{-}{-}h(n-1)_{n-1}a_n$$
$$+\ \Sigma h(1)_1{-}{-}{-}h(n-1)_{n-1}b_n+\Sigma g(1)_1{-}{-}{-}g(n-1)_{n-1}x_n+\Sigma h(1)_1{-}{-}{-}h(n-1)_{n-1}x_n$$
$$+\ x_1{-}{-}{-}x_{n-1}a_n+x_1{-}{-}{-}x_{n-1}b_n\ .$$

The third, sixth, and seventh terms of this sum give us $\Sigma h(1)_1{-}{-}{-}h(n)_n$.

Since $\Sigma h(1)_1{-}{-}{-}h(n-1)_{n-1}b_n = \prod_{i=1}^{n-1}(\lambda a_i+x_i)b_n - x_1{-}{-}{-}x_{n-1}b_n$ the remainingterms give us $\Sigma g(1)_1{-}{-}{-}g(n)_n$. This completes the proof.

Taking into account the remarks preceeding Proposition A.1, equation iv) follows immediately from A.2.

In [1] Barr conctructs the extension which realizes an unobstructed p as a certain coequalizer. In the notation of our section II, his diagram on page 365 would look like:

$$\begin{tikzcd}
 & 0 \arrow[d] & \\
 & (A,\ \alpha_1) \arrow[r, "A"] \arrow[d, "i_1"] & (A,\ \alpha_1) \arrow[d] \\
(T^2R,\ \mu TR) \arrow[r, shift left, "O\mathrm{x}\mu R"] \arrow[r, shift right, "p_1\cdot\nu_1\cdot Tu_1\mathrm{x}T\xi_1"'] \arrow[d, "T^2R"'] & (A\mathrm{x}TR,-) \arrow[r, "(p_1+p_1\cdot u_1\cdot p_2)\mathrm{x}\xi_1\cdot p_2"] \arrow[d, "p_2"] & (A\mathrm{x}R,\ \zeta_1) \arrow[d, "p_2"] \\
(T^2R,\ \mu TR) \arrow[r, shift left, "\mu R"] \arrow[r, shift right, "T\xi_1"'] & (TR,\ \mu R) \arrow[r, "\xi_1"] \arrow[d] & (R,\ \xi_1) \arrow[d] \\
 & 0 & 0
\end{tikzcd}$$

He uses the coequalizer $(p_1+p_1.u_1.p_2)x\xi_1.p_2$ to define the algebra which gives the extension, and then must make some rather tedious computations to verify that all requirements are met. One knows that if an extension exists, then its underlying set will have to be AxR: the only question is how the algebra structure on AxR is "twisted". By passing to the equivalent category of T-algebras it becomes clear exactly how the cocycle should be used to produce this twisted structure. All of this was first noticed by Beck in the case of singular extensions [2] . At any rate, the "globalization" of Barr's results seems to require that we pass to $(\mathrm{Sets}^{|X|})^T_G$.

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